



Final Examination

MATH-337
COMBINATORIAL NUMBER THEORY
FALL SEMESTER 2024

22 JANUARY, 2025

First name: _____

Last name: _____

EPFL email: _____@epfl.ch

Instructions

- ◆ This examination consists of 9 questions, out of which you have to answer 6. You can choose yourself which 6.
- ◆ Do not submit solutions for more than 6 questions. If you submit solutions to more, then only the ones with the least score will count towards your final grade.
- ◆ You have three hours (180 minutes) to complete this exam.
- ◆ The use of books, notes, calculators, computers, tablets or phones is prohibited.
- ◆ Write legibly and show all of your work. Unsupported answers may not earn credit. Cross out any work that you do not wish to have scored.
- ◆ Write your solutions only on the provided exam paper within the framed area. Do not use your own paper. Work written outside the margin may not be scored.
- ◆ Avoid combining solutions for multiple problems on one paper. Instead, start a new sheet for each problem's solution.
- ◆ You are permitted to use results from the course without proving them if you state and apply them correctly.
- ◆ Each question on this exam is graded out of 8 points using the same rubric as we did for homework assignments: mathematical correctness (worth 5 points) and proof-writing quality (worth 3 points). The maximal amount of points that you can score on this exam is 48.

Grading Table

(for examiner use only)

Question	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
Total:	48	

1. (8 points) State and prove the Compactness Theorem for finite colorings.
2. (8 points) Let \mathcal{P} be an upward closed family of subsets of a set X . Prove that \mathcal{P} is partition regular if and only if \mathcal{P}^* is closed under finite intersections.
3. (8 points) State and prove Rado's Theorem for single equations. (You may use Brauer's theorem without proving it.)
4. (8 points) State Ramsey's Theorem for graphs, state Schur's Theorem, and then prove that the former implies the latter.
5. (8 points) Let $N \in \mathbb{N}$ and for $A \subseteq \mathbb{Z}_N$ and $x \in \mathbb{Z}_N$ define $r(x) = |\{(a_1, a_2) \in A \times A : a_1 + a_2 = x\}|$. Prove that

$$r(x) = \frac{|A|^2}{N} + N \sum_{\xi \in \mathbb{Z}_N \setminus \{0\}} e\left(\frac{-x\xi}{N}\right) \hat{A}(\xi)^2.$$
6. (8 points) This problem has two parts.
 - (a) Show that for any finite coloring of \mathbb{N} there exist $x, y, z \in \mathbb{N} \setminus \{1\}$ of the same color satisfying the equation $xy = z^2$.
 - (b) Show that any multiplicatively syndetic set $S \subseteq \mathbb{N}$ contains a solution to the equation $xy = z^2$.
7. (8 points) Let $D \subseteq \mathbb{N}$ be a subset of the natural numbers with full density, that is, $d(D) = 1$. Prove that D is an IP-set.
8. (8 points) Let $S \subseteq \mathbb{N}$. Prove that the following two statements are equivalent:
 - (i) The set S is syndetic.
 - (ii) For every ultrafilter $p \in \beta\mathbb{N}$ the set $S - p$ is syndetic.
9. (8 points) Prove that for any thick set $T \subseteq \mathbb{N}$ and any $h \in \mathbb{N}$ the set $T \cap (T-1) \cap \dots \cap (T-h)$ is thick.